ITEP-TH-15/04

Hochschild Cohomology versus De Rham Cohomology without Formality Theorems.

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Abstract

We exploit the Fedosov-Weinstein-Xu (FWX) resolution proposed in q-alg/9709043 to establish an isomorphism between the ring of Hochschild cohomology of the quantum algebra of functions on a symplectic manifold M and the ring $H^{\bullet}(M, \mathbb{C}((\hbar)))$ of De Rham cohomology of M with the coefficient field $\mathbb{C}((\hbar))$ without making use of any version of the formality theorem. We also show that the Gerstenhaber bracket induced on $H^{\bullet}(M, \mathbb{C}((\hbar)))$ via the isomorphism is vanishing. We discuss equivariant properties of the isomorphism and propose an analogue of this statement in an algebraic geometric setting.

1 Introduction

Although Kontsevich's formality theorem [22],[23], [31] and its generalizations [5],[7], [9], [11], [12], [16], [29], [30], [32], [33] give immediate solutions of various problems of deformation quantization, the complicated technique of homotopy algebraic structures is not required for many of these problems. In fact, a number of examples show that for many problems of deformation quantization this technique can be replaced by simpler arguments [4], [10], [13], [14], [20], [25], [26], [27], [28], [34], [35].

In paper [34] by A. Weinstein and P. Xu the authors proposed a resolution of the vector space of local Hochschild cochains of the ring of quantum functions on a symplectic manifold M. They used this resolution to prove that the graded vector space of Hochschild cohomology of the ring of quantum functions on a symplectic manifold is isomorphic to the graded vector space $H^{\bullet}(M, \mathbb{C}((\hbar)))$ of De Rham cohomology of M with the coefficients in the field $\mathbb{C}((\hbar))$ of formal Laurent power series. In our paper we use this resolution to show that the above cohomology spaces are isomorphic as rings. In principle, the compatibility of

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Kontsevich's quasi-isomorphism [22] with the cup-product in Hochschild cohomology implies an analogous statement for a general Poisson manifold. However, it is instructive to see that in the symplectic case, the result can be proven by a much simpler argument. From our considerations it will also be transparent that the Gerstenhaber bracket induced on De Rham cohomology via the above isomorphism with Hochschild cohomology is vanishing.

A dual version of this assertion was proven in paper [26] by Nest and Tsygan on the algebraic index theorem. Namely, in [26] the authors describe Hochschild and cyclic homology of the algebra of quantum functions with compact support on an arbitrary symplectic manifold using the spectral sequence associated to the $\hbar$-adic filtration.

In this context, it is worth mentioning paper [6], in which similar results were obtained for Hochschild and cyclic homology of the ring of pseudo-differential symbols on an arbitrary smooth manifold.

The structure of this paper is as follows. In the next section we formulate the main results of the paper (See theorems 1 and 2). In the third section we remind the Fedosov deformation quantization, recall the Fedosov-Weinstein-Xu resolution of the algebra of local Hochschild cochains of the quantum ring of functions on a symplectic manifold, and prove theorems 1 and 2 formulated in the previous section. In the concluding section we discuss applications and the variations of theorems 1 and 2 . In Appendices A and B at the end of the paper we recall algebraic structures on Hochschild cochains and propose an equivariant homotopy operator for the cohomological Hochschild complex of the formal Weyl algebra.

Throughout the paper we assume the summation over repeated indices. We omit symbol $\wedge$ referring to a local basis of exterior forms as if we thought of dx^i's as anti-commuting variables. We denote by d the De Rham differential. By a nilpotent linear operator we always mean an operator whose second power is vanishing.

2 Algebra of local Hochschild cochains of the ring of quantum functions on a symplectic manifold.

Let M be an even dimensional smooth manifold endowed with a symplectic form $\omega = \omega_{ij}(x) dx^i \wedge dx^j$. Here i, j run from 1 to $2n = dim M$ and x^i denote local coordinates. Let $\omega^{ij}(x)\partial_{x^i} \wedge \partial_{x^j}$ denote the respective Poisson tensor

$$\omega^{ik}(x)\omega_{kj}(x) = \delta^i_j \, .$$

According to the standard terminology of deformation quantization [2], [3] a star-product on the symplectic manifold M is an associative non-commutative $\mathbb{C}((\hbar))$-linear product in $C^\infty(M)((\hbar))$ given by the formal power series of bidifferential operators S_k

$$a * b = ab + \sum_{k=1}^{\infty} \hbar^k S_k(a,b) \, , \qquad a, b \in C^\infty(M) \tag{2.1}$$

and such that

$$a * b - b * a = \hbar\{a, b\} \; mod \; \hbar^2 \, , \qquad a * 1 = 1 * a = a.$$

where $\{,\}$ denotes the Poisson bracket associated to the symplectic structure on M .

Furthermore, two star-products $*$ and $*'$ are called equivalent if there exists a formal series

$$Q = I + \sum_{k=1}^{\infty} \hbar^k Q_k$$

of differential operators Q_k on M such that

$$Q(a * b) = (Qa) *' (Qb)\,, \qquad \forall\; a, b \in C^\infty(M)((\hbar))\,. \tag{2.2}$$

We denote by $\mathbb{A}$ the algebra $C^\infty(M)((\hbar))$ of formal Laurent power series of complex valued smooth functions on M with the multiplication (2.1).

By definition the vector space $C^k_{loc}(\mathbb{A})$ of *local Hochschild k-cochains* of the algebra $\mathbb{A}$ is the subspace of $C^k_{loc}(\mathbb{A}) \subset Hom_{\mathbb{C}((\hbar))}(\mathbb{A}^{\otimes k}, \mathbb{A})$ of $\mathbb{C}((\hbar))$-linear homomorphisms from $\mathbb{A}^{\otimes k}$ to $\mathbb{A}$ preserving supports of functions. In [8] it is shown that any polylinear map $L \in Hom_{\mathbb{C}}((C^\infty(M))^{\otimes k}, C^\infty(M))$ which preserves supports of functions can be locally represented as a polydifferential operator[2]. Thus any element $P \in C^k_{loc}(\mathbb{A})$ can be defined locally as the following formal series of k-differential operators

$$P(a_1, \dots, a_k) = \sum_{m \in \mathbb{Z}} \hbar^m P_m(a_1, \dots, a_k)\,, \tag{2.3}$$

where the summation in m is bounded below.

It is not hard to see that the Hochschild differential ∂ (A.3), the cup-product (A.4) and the Gerstenhaber bracket (A.6) preserve locality of Hochschild cochains and therefore are well-defined on $C^\bullet_{loc}(\mathbb{A})$. In what follows by the ring of Hochschild cohomology of $\mathbb{A}$ we mean the vector space

$$HH^\bullet_{loc}(\mathbb{A}) = H(C^\bullet_{loc}(\mathbb{A}), \partial)$$

with the multiplication induced by (A.4).

We have to mention that since $\mathbb{A}$ is a deformation of the algebra of smooth function on M the complex of all Hochschild cochains of $\mathbb{A}$ is a rather intractable object. In particular, Hochschild cohomology $HH^k(A)$ for $k > 1$ is not known even for the ordinary commutative algebra of smooth functions on $\mathbb{R}^n$.

Recall that given a symplectic torsion free connection ∇ and a Fedosov representative $\Omega_F \in \dfrac{1}{\hbar}\Omega^2(M)[[\hbar]]$ (3.24) one can construct an algebra $\mathbb{A}_{\nabla,\Omega_F}$ which quantizes the symplectic manifold M in the sense of (2.1). It is well-known (see theorem 5) that for different Fedosov representatives Ω_F the algebras $\mathbb{A}_{\nabla,\Omega_F}$ exhaust all equivalence classes of deformation quantizations of M. In this paper we refer to the triple (M, ∇, Ω_F) consisting of a symplectic manifold M, a symplectic torsion free connection ∇ and a Fedosov representative Ω_F (3.24) as *Fedosov data*.

The main results of the paper are formulated in the following two theorems

Theorem 1 *For the algebra $\mathbb{A} = (C^\infty(M)((\hbar)), *)$ of quantum functions on a symplectic manifold M the ring of Hochschild cohomology*

$$HH^\bullet_{loc}(\mathbb{A})$$

[2] If the manifold M is compact the word “locally” can be omitted.

is isomorphic to the ring of De Rham cohomology

$$H_{DR}(M) \otimes \mathbb{C}((\hbar))\,,$$

with coefficients in the field $\mathbb{C}((\hbar))$ of formal Laurent power series. The Lie bracket induced on $H^\bullet_{DR}(M) \otimes \mathbb{C}((\hbar))$ via this isomorphism is vanishing.

Theorem 2 *To the Fedosov data (M, ∇, Ω_F) one can naturally assign a differential graded (DG) associative algebra $K^\bullet_{\nabla,\Omega_F}$ and a pair of embeddings*

$$\mathfrak{E}_{\nabla,\Omega_F} \,:\, \Omega^\bullet(M)((\hbar)) \hookrightarrow K^\bullet_{\nabla,\Omega_F}\,, \tag{2.4}$$

$$\mathfrak{D}_{\nabla,\Omega_F} \,:\, C^\bullet_{loc}(\mathbb{A}_{\nabla,\Omega_F}) \hookrightarrow K^\bullet_{\nabla,\Omega_F}\,, \tag{2.5}$$

which are quasi-isomorphisms of the corresponding DG associative algebras. If g is a diffeomorphism of M then the corresponding embeddings $\mathfrak{D}_{\nabla,\Omega_F}$, $\mathfrak{E}_{\nabla,\Omega_F}$, $\mathfrak{D}_{g_*\nabla,g_*\Omega_F}$, and $\mathfrak{E}_{g_*\nabla,g_*\Omega_F}$ fit into the commutative diagram

$$\begin{array}{ccccc} C^\bullet_{loc}(\mathbb{A}_{\nabla,\Omega_F}) & \stackrel{\mathfrak{D}_{\nabla,\Omega_F}}{\longrightarrow} & K^\bullet_{\nabla,\Omega_F} & \stackrel{\mathfrak{E}_{\nabla,\Omega_F}}{\longleftarrow} & \Omega^\bullet(M) \otimes \mathbb{C}((\hbar)) \\ \downarrow^{g_*} & & \downarrow^{g_*} & & \downarrow^{g_*} \\ C^\bullet_{loc}(\mathbb{A}_{g_*\nabla,\,g_*\Omega_F}) & \stackrel{\mathfrak{D}_{g_*\nabla,\,g_*\Omega_F}}{\longrightarrow} & K^\bullet_{g_*\nabla,\,g_*\Omega_F} & \stackrel{\mathfrak{E}_{g_*\nabla,\,g_*\Omega_F}}{\longleftarrow} & \Omega^\bullet(M) \otimes \mathbb{C}((\hbar))\,. \end{array} \tag{2.6}$$

Proofs of the above statements are given in the next section. The proofs are based on the use of the Fedosov-Weinstein-Xu (FWX) resolution [34] of the vector space of local Hochschild cochains of $\mathbb{A}_{\nabla,\Omega_F}$. An interesting corollary and a couple of variations of theorems 1 and 2 are discussed in the concluding section of the paper.

Remark. We would like to mention that the triviality of the operation induced by the Gerstenhaber bracket on the Hochschild cohomology can be viewed as a generalization of the familiar fact that the Lie bracket of symplectic vector fields is a Hamiltonian vector field.

3 FWX resolution of the algebra of local Hochschild cochains

In this section we recall the construction of the Fedosov-Weinstein-Xu (FWX) resolution [34] of the algebra of local Hochschild cochains of the quantum ring of functions on a symplectic manifold. A classical analogue of this resolution was used in papers [11] and [12] to prove the formality theorems for Hochschild (co)chains of the algebra of functions on an arbitrary smooth manifold. We start with a brief reminder of the Fedosov deformation quantization.

3.1 Reminder of the Fedosov deformation quantization

As above, M stands for an even dimensional smooth manifold endowed with a symplectic form $\omega = \omega_{ij}(x)dx^i \wedge dx^j$ and $\omega^{ij}(x)\partial_{x^i} \wedge \partial_{x^j}$ denotes the respective Poisson tensor

$$\omega^{ik}(x)\omega_{kj}(x) = \delta^i_j\,.$$

Following Fedosov [13] we introduce the Weyl algebra bundle over the manifold M. Sections of this bundle are the following formal sums

$$a = a(x, \hbar, y) = \sum_{p\geq 0,\, k\in\mathbb{Z}} \hbar^k a_{k;i_1\dots i_p}(x) y^{i_1}\dots y^{i_p}\,, \tag{3.1}$$

where the summation in k is bounded below, $a_{k;i_1\dots i_p}(x)$ are symmetric covariant $\mathbb{C}$-valued tensors, and y^i are fiber coordinates on the tangent bundle TM. An associative multiplication of the sections (3.1) is defined with the help of the Poisson tensor ω^{ij} as follows

$$a\circ b(x,\hbar,y) = exp\left(\frac{\hbar}{2}\omega^{ij}(x)\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(x,\hbar,y)\, b(x,\hbar,z)|_{z=y}\,. \tag{3.2}$$

The bundle $\mathcal{W}$ viewed as a sheaf of algebras over $\mathbb{C}$ is naturally filtered with respect to the degree of monomials $2[\hbar]+[y]$ where $[\hbar]$ is a degree in $\hbar$ and $[y]$ is a degree in y

$$\begin{gathered}\dots\subset\mathcal{W}^1\subset\mathcal{W}^0\subset\mathcal{W}^{-1}\dots\subset\mathcal{W}\,,\\ \Gamma(X,\mathcal{W}^m) = \{a = \sum_{2k+p\geq m}\hbar^k a_{k;i_1\dots i_p}(x)y^{i_1}\dots y^{i_p}\}\end{gathered} \tag{3.3}$$

This filtration defines the $2[\hbar]+[y]$-adic topology in the algebra of section $\Gamma(X,\mathcal{W})$ over any open subset $X\subset M$.

Let us remark that the space $\Omega^\bullet(M,\mathcal{W})$ of smooth exterior forms with values in $\mathcal{W}$ is naturally a graded associative algebra with the product induced by (3.2) and the following graded commutator

$$[a,b] = a\circ b - (-)^{q_a q_b} b\circ a\,,$$

where q_a and q_b are exterior degrees of a and b, respectively. The filtration of $\mathcal{W}$ (3.3) gives us a filtration of the algebra $\Omega^\bullet(M,\mathcal{W})$

$$\dots\subset\Omega(M,\mathcal{W}^1)\subset\Omega(M,\mathcal{W}^0)\subset\Omega(M,\mathcal{W}^{-1})\dots\subset\Omega(M,\mathcal{W})\,, \tag{3.4}$$

which similarly defines the $2[\hbar]+[y]$-adic topology in the algebra $\Omega(M,\mathcal{W})$ of exterior forms.

Following Fedosov [13] we introduce the following derivation of the algebra $\Omega(M,\mathcal{W})$

$$\delta = dx^i\frac{\partial}{\partial y^i}\;:\;\Omega^\bullet(M,\mathcal{W})\mapsto\Omega^{\bullet+1}(M,\mathcal{W})\,,\qquad \delta^2=0\,. \tag{3.5}$$

The cohomology of the differential δ are described by the following

Proposition 1

$$H^0(\Omega(M,\mathcal{W}),\delta) = C^\infty(M)((\hbar))\,,\qquad H^{\geq 1}(\Omega(M,\mathcal{W}),\delta)=0\,.$$

Proof. It is not hard to guess the map

$$\sigma\;:\;\Omega(M,\mathcal{W})\mapsto C^\infty(M)((\hbar))$$

from the complex $(\Omega(M,\mathcal{W}),\delta)$ onto its subcomplex $(C^\infty(M)((\hbar)),0)$ and a contracting homotopy δ^{-1}

$$\delta^{-1}\;:\;\Omega^\bullet(M,\mathcal{W})\mapsto\Omega^{\bullet-1}(M,\mathcal{W})\,,$$

between the map σ and the identity map. Namely,

$$\sigma(a) = a\Big|_{y=0,\ dx=0}, \qquad a \in \Omega^\bullet(M,\mathcal{W}). \tag{3.6}$$

and

$$\delta^{-1}a = y^k i\left(\frac{\partial}{\partial x^k}\right)\int_0^1 a(x,\hbar,ty,tdx)\frac{dt}{t}, \tag{3.7}$$

where $i(\partial/\partial x^k)$ denotes the contraction of an exterior form with the vector field $\partial/\partial x^k$, and δ^{-1} is extended to $\Gamma(\mathcal{W})$ by zero.

The desired contracting property

$$a = \sigma(a) + \delta\delta^{-1}a + \delta^{-1}\delta a\,, \qquad \forall\ a \in \Omega(M,\mathcal{W}) \tag{3.8}$$

can be checked by a straightforward computation. □
We remark that the operator δ^{-1} turns out to be nilpotent

$$(\delta^{-1})^2 = 0\,. \tag{3.9}$$

Recall that on any symplectic manifold there exists an affine torsion-free connection ∇_i which is compatible with the symplectic structure [18]

$$\nabla_i\omega_{jk}(x) = 0\,.$$

This connection allows us to define the following derivation of the algebra $\Omega(M,\mathcal{W})$

$$\nabla = dx^i\frac{\partial}{\partial x^i} - dx^i\Gamma^j_{ik}(x)y^k\frac{\partial}{\partial y^j}\ :\ \Omega^\bullet(M,\mathcal{W}) \mapsto \Omega^{\bullet+1}(M,\mathcal{W})\,, \tag{3.10}$$

where $\Gamma^j_{ik}(x)$ are the respective Christoffel symbols. The compatibility of the connection with the symplectic structure implies that (3.10) is a derivation of the product (3.2) .

Since the connection ∇_i is torsion-free the derivation ∇ anti-commutes with δ

$$\nabla\delta + \delta\nabla\,. \tag{3.11}$$

In general the derivation ∇ is not nilpotent as δ. Instead we have the following expression for ∇^2

$$\nabla^2 a = \frac{1}{\hbar}[R,a]\ :\ \Omega^\bullet(M,\mathcal{W}) \mapsto \Omega^{\bullet+2}(M,\mathcal{W})\,, \tag{3.12}$$

where

$$R = -\frac{1}{4}dx^i dx^j \omega_{km}(R_{ij})^m_l(x)y^k y^l\,,$$

and $(R_{ij})^k_l(x)$ is the standard Riemann curvature tensor of the connection ∇_i.

Let us consider the following derivation of the algebra $\Omega(M,\mathcal{W})$ (connection on the bundle $\mathcal{W}$)

$$D = \nabla - \delta + \frac{1}{\hbar}[\,r,\,\bullet\,]\,, \tag{3.13}$$

where r is a smooth 1-form with values in $\mathcal{W}^3$. Using equations (3.11) and (3.12) we get that for any $a \in \Omega(M, \mathcal{W})$

$$D^2 a = \frac{1}{\hbar}[R - \delta r + \nabla r + \frac{1}{\hbar} r \circ r, a] \,. \tag{3.14}$$

Since the center of the Weyl algebra is $\mathbb{C}((\hbar)) \subset \mathbb{C}[[y^1, \dots, y^{2n}]]((\hbar))$ the derivation D is nilpotent (the connection (3.13) is flat) if and only if

$$R - \delta r + \nabla r + \frac{1}{\hbar} r \circ r \in \Omega^2(M)((\hbar)) \,. \tag{3.15}$$

Furthermore, since $r \in \Omega^1(M, \mathcal{W}^3)$ we have

$$R - \delta r + \nabla r + \frac{1}{\hbar} r \circ r \in \hbar \Omega^2(M)[[\hbar]] \,. \tag{3.16}$$

Let us say that

$$R - \delta r + \nabla r + \frac{1}{\hbar} r \circ r = \sum_{k \geq 1} \hbar^k \omega_k \,. \tag{3.17}$$

Using the properties of ∇ and δ and the Bianchi identities for the Riemann curvature tensor $(R_{ij})^k_l$

$$\delta R = 0 \,, \qquad \nabla R = 0 \,.$$

we derive that

$$D(R - \delta r + \nabla r + \frac{1}{\hbar} r \circ r) = 0 \,.$$

The latter is equivalent to the fact that all the forms ω_k in the right hand side of (3.17) are closed with respect to the De Rham differential d.

It turns out that for any series of closed forms

$$\Omega = \sum_{k \geq 1} \hbar^k \omega_k \tag{3.18}$$

one can construct an element $r \in \Omega^1(M, \mathcal{W}^3)$ satisfying (3.17). Namely, theorem 5.3.3 in [14] implies that

Theorem 3 (Fedosov) *For any formal series (3.18) of closed forms on M one can construct an element*

$$r = \sum_{k \geq 0,\, p \geq 1} dx^l \hbar^k r_{k;l,i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \in \Omega^1(M, \mathcal{W}^3) \tag{3.19}$$

satisfying (3.17) and the condition $\delta^{-1} r = 0$. The operation

$$D = \nabla - \delta + \frac{1}{\hbar}[\, r, \bullet \,] \tag{3.20}$$

associated to the element (3.19) is a nilpotent derivation of the algebra $\Omega(M, \mathcal{W})$.

Remark. The element r in the above theorem can be obtained by iterating the following equation

$$r = \delta^{-1}(R - \Omega) + \delta^{-1}(\nabla r + \frac{1}{\hbar} r \circ r) \,. \tag{3.21}$$

Since the operator ∇ preserves filtration (3.4) and δ^{-1} raises it by 1, the iteration of (3.21) converges in the topology (3.4) and defines the unique element $r \in \Omega^1(M, \mathcal{W}^3)$.

Furthermore, the respective generalization of theorem 5.2.4 in [14] says that

Theorem 4 (Fedosov) *Iterating the equation*

$$\tau(a) = a + \delta^{-1}(\nabla \tau(a) + \frac{1}{\hbar}[r, \tau(a)]) \tag{3.22}$$

for $a \in C^\infty(M)((\hbar))$ *one constructs an isomorphism*

$$\tau \,:\, C^\infty(M)((\hbar)) \mapsto ker\, D \cap \Gamma(M, \mathcal{W})$$

from the vector space $C^\infty(M)((\hbar))$ *to the vector space of horizontal sections* $Z^0(\Omega(M, \mathcal{W}), D)$ *of the connection (3.20). For any* $a \in C^\infty(M)((\hbar))$

$$\sigma(\tau(a)) = a$$

and the multiplication on $C^\infty(M)((\hbar))$ *induced by the* $\circ$*-product (3.2) via the isomorphism* τ

$$a * b = \sigma(\tau(a) \circ \tau(b)), \qquad a, b \in C^\infty(M)((\hbar)) \tag{3.23}$$

is a star-product associated to the Poisson bracket ω^{ij} *.* $\square$

Notice that since the operator ∇ preserves the filtration (3.4) and δ^{-1} raises it by 1, the iteration of (3.22) converges in the topology (3.4) and define the unique element $\tau(a) \in Z^0(\Omega(M, \mathcal{W}), D)$ for any $a \in C^\infty(M)((\hbar))$.

One can easily observe that the series (3.18) of closed forms enters the construction of the star-product (3.23). In fact it is not hard to show that the equivalence class of the star-product (3.23) depends only on the cohomology class of the series (3.18). In other words, if $*$ and $*'$ are Fedosov star-products corresponding to series Ω and Ω' representing the same cohomology class in $H^2(M, \mathbb{C})((\hbar))$ then $*$ is equivalent to $*'$ in the sense of (2.2). Furthermore, if two series Ω and Ω' define distinct cohomology classes in $H^2(M, \mathbb{C})((\hbar))$ then the corresponding Fedosov star-products $*$ and $*'$ are not equivalent.

The following result of P. Xu [35] shows that the above construction allows us to get a star-product from any equivalence class of the star-products on M

Theorem 5 (P. Xu, [35]) *Any star-product on a symplectic (smooth real) manifold* M *is equivalent to some Fedosov star-product.* $\square$

If a star-product $*$ on M is equivalent to the Fedosov star-product corresponding to the series (3.18) the cohomology class of the combination

$$\Omega_F = \frac{1}{\hbar}(-\omega + \Omega) \in \frac{1}{\hbar} Z^2_d(\Omega(M)[[\hbar]]) \tag{3.24}$$

is called the Fedosov class of a star-product $*$. We refer to Ω_F entering the construction of the star-product as *the Fedosov representative.*

3.2 Double complex of fiberwise Hochschild cochains.

In this section we denote by $\mathbb{A}$ the algebra of functions $C^\infty(M)((\hbar))$ with the star-product $*$ (3.23) associated to the Fedosov data (M, ∇, Ω_F).

We now turn to the definition of formal fiberwise Hochschild cochains on $\Gamma(\mathcal{W})$

Definition 1 *A bundle $\mathcal{C}^k$ of formal fiberwise Hochschild cochains of degree k is a bundle over M whose sections are $C^\infty(M)$-polylinear maps $\mathfrak{P} : \bigotimes^k \Gamma(\mathcal{W}) \mapsto \Gamma(\mathcal{W})$ continuous in the adic topology (3.3).*

Any such map $\mathfrak{P} \in \Gamma(\mathcal{C}^k)$ can be uniquely represented in the form of the following formal series

$$\mathfrak{P} = \sum_{\alpha_1 \dots \alpha_k} \sum_{m,p=0}^{\infty} \hbar^m \mathfrak{P}^{\alpha_1 \dots \alpha_k}_{m, i_1 \dots i_p}(x) y^{i_1} \dots y^{i_p} \frac{\partial}{\partial y^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{\alpha_k}} \,, \tag{3.25}$$

where the summation in m is bounded below, α's are multi-indices $\alpha = j_1 \dots j_l$,

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial}{\partial y^{j_1}} \dots \frac{\partial}{\partial y^{j_l}} \,,$$

and the tensors $\mathfrak{P}^{\alpha_1 \dots \alpha_k}_{m; i_1 \dots i_p}(x)$ are symmetric in covariant indices $i_1, \dots, i_p$.

Extending the above definition by allowing the cochains to be inhomogeneous we define the total bundle $\mathcal{C}$ of formal fiberwise Hochschild cochains as a direct sum

$$\mathcal{C} = \bigoplus_{k=0}^{\infty} \mathcal{C}^k \,, \qquad \mathcal{C}^0 = \mathcal{W} \,. \tag{3.26}$$

The space $\Omega(M, \mathcal{C})$ of smooth exterior forms with values in the bundle $\mathcal{C}$ acquires a natural associative product induced by the fiberwise cup-product (A.4) in the space of Hochschild cochains of the formal Weyl algebra. Here we also call this product a cup-product. So, given $\mathfrak{P}_1 \in \Omega(M, \mathcal{C}^{k_1})$ and $\mathfrak{P}_2 \in \Omega(M, \mathcal{C}^{k_2})$ their cup-product is defined by

$$\mathfrak{P}_1 \cup \mathfrak{P}_2(a_1, a_2, \dots, a_{k_1+k_2}) = \mathfrak{P}_1(a_1, \dots, a_{k_1}) \circ \mathfrak{P}_2(a_{k_1+1}, \dots, a_{k_1+k_2}) \,, \tag{3.27}$$

where a_i are arbitrary smooth sections of the Weyl algebra bundle $\mathcal{W}$.

The space $\Omega(M, \mathcal{C})$ also acquires a graded Lie algebra structure induced by the fiberwise Gerstenhaber bracket (A.6). For two homogeneous elements $\mathfrak{P}_1 \in \Omega(M, \mathcal{C}^{k_1+1})$ and $\mathfrak{P}_2 \in \Omega(M, \mathcal{C}^{k_2+1})$ the bracket is defined as follows

$$\begin{gathered}
[\mathfrak{P}_1, \mathfrak{P}_2]_G(a_0, \dots, a_{k_1+k_2}) = \\
\sum_{i=0}^{k_1} (-)^{ik_2} \mathfrak{P}_1(a_0, \dots, \mathfrak{P}_2(a_i, \dots, a_{i+k_2}), \dots, a_{k_1+k_2}) \\
-(-)^{k_1 k_2} (1 \leftrightarrow 2) \,, \qquad a_j \in \Gamma(M, \mathcal{W}) \,.
\end{gathered} \tag{3.28}$$

The fiberwise $\circ$ product in $\Gamma(\mathcal{W})$ gives us the fiberwise Hochschild differential (A.3)

$$(\partial \mathfrak{P})(a_0, a_1, \dots, a_k) = (-)^q (a_0 \circ \mathfrak{P}(a_1, \dots, a_k) - \mathfrak{P}(a_0 \circ a_1, a_2, \dots, a_k) + \dots + \tag{3.29}$$

$$(-1)^k\mathfrak{P}(a_0,\dots,a_{k-2},a_{k-1}\circ a_k)+(-1)^{k+1}\mathfrak{P}(a_0,a_1,\dots,a_{k-1})\circ a_k)\,,$$
$$\partial\,:\,\Omega^q(M,\mathcal{C}^k)\mapsto\Omega^q(M,\mathcal{C}^{k+1})\,,$$

which is a derivation of both the cup-product (3.27) and the Gerstenhaber bracket (3.28) by (A.5) (A.8).

The Fedosov differential (3.13) can be naturally extended to the vector space $\Omega(M,\mathcal{C})$ via the formula

$$(D\mathfrak{P})(a_1,\dots,a_k)=$$

$$D\mathfrak{P}(a_1,\dots,a_k)-(-)^q(\mathfrak{P}(Da_1,a_2\dots,a_k)+\dots+\mathfrak{P}(a_1,a_2\dots,Da_k))\,,\tag{3.30}$$

where $\mathfrak{P}\in\Omega^q(M,\mathcal{C}^k)$ and a_i are arbitrary smooth sections of the Weyl algebra bundle $\mathcal{W}$. It is not hard to see that the differential (3.30) is a derivation of the cup-product (3.27) and the Gerstenhaber bracket (3.28). The differential (3.30) also anti-commutes with the fiberwise Hochschild differential ∂

$$D\,\partial+\partial\,D=0$$

since D is a derivation of the $\circ$-product (3.2).

Thus we arrive at the double complex $(\Omega^\bullet(M,\mathcal{C}^\bullet),D,\partial)$, the total space of which is a differential graded associative algebra (DGAA) and also a differential graded Lie algebra (DGLA).

Due to propositions 8 and 9 given in Appendix B the fiberwise Hochschild differential ∂ has vanishing higher cohomology

$$H^{\geq 1}(\Omega(M,\mathcal{C}^\bullet),\partial)=0\,.\tag{3.31}$$

Therefore for any $q\geq 0$ the natural embedding

$$\mathfrak{E}^q\,:\,\Omega^q(M)((\hbar))\hookrightarrow\Omega^q(M,\mathcal{C}^0)\tag{3.32}$$

induces a quasi-isomorphism of complexes $(\Omega^q(M)((\hbar)),0)$ and $(\Omega^q(M,\mathcal{C}^\bullet),\partial)$. Furthermore, since the Fedosov differential restricted to $\Omega(M)((\hbar))$ coincides with the De Rham differential d the inclusion $\mathfrak{E}\,:\,\Omega(M)\hookrightarrow\Omega(M,\mathcal{C}^0)$ also induces a morphism from the De Rham complex $(\Omega(M)((\hbar)),d)$ to the total complex $(\Omega(M,\mathcal{C}^\bullet),D+\partial)$. One can easily see that $\mathfrak{E}$ is compatible with the cup-products and therefore $\mathfrak{E}$ is a morphism of DG associative algebras

$$\mathfrak{E}\,:\,(\Omega(M)((\hbar)),d,\wedge)\hookrightarrow(\Omega(M,\mathcal{C}),D+\partial,\cup)\,.\tag{3.33}$$

Due to (3.31) and (3.32) we have that

Proposition 2 *The spectral sequence of the double complex* $(\Omega^\bullet(M,\mathcal{C}^\bullet),D,\partial)$, *associated to the filtration by the degree of fiberwise Hochschild cochains degenerates at* E_1 *and the map* $\mathfrak{E}$ *(3.33) is a quasi-isomorphism of DG associative algebras.* $\square$

We will prove the desired statements of theorems 1 and 2 by computing the total cohomology of double complex $(\Omega^\bullet(M,\mathcal{C}^\bullet),D,\partial)$ using the spectral sequence associated to the filtration by the exterior degree in $\Omega^\bullet(M,\mathcal{C}^\bullet)$. But before doing this, we need to extend the operations introduced on the vector space $\Omega(M,\mathcal{W})$ in the previous subsection to the vector space $\Omega(M,\mathcal{C})$. A proof of the following proposition is straightforward

Proposition 3

1. *The nilpotent derivation* δ *(3.5) and the covariant derivative* ∇ *(3.10) are extended to* $\Omega(M,\mathcal{C})$ *in the following natural manner*

$$\delta\mathfrak{P} = [dx^i \frac{\partial}{\partial y^i}, \mathfrak{P}]_G \,, \qquad \nabla\mathfrak{P} = dx^i \frac{\partial}{\partial x^i}\mathfrak{P} - [\, dx^i \Gamma^j_{ik} y^k \frac{\partial}{\partial y^j} \,, \mathfrak{P}]_G \,, \tag{3.34}$$

where $dx^i \Gamma^j_{ik} y^k \frac{\partial}{\partial y^j}$ *is viewed locally as an element of* $\Omega^1(\,\bullet\,, \mathcal{C}^1)$ *. With this definition* ∇ *is globally defined,* δ *is nilpotent and equation (3.11) holds.*

2. *The component-wise extension of the map* σ *(3.6)*

$$\sigma\mathfrak{P} = \mathfrak{P}\Big|_{y^i = dx^i = 0} \;:\; \Omega(M,\mathcal{C}) \mapsto Z^0_\delta(\Omega(M,\mathcal{C})) \tag{3.35}$$

defines a projection onto the kernel $Z^0_\delta(\Omega(M,\mathcal{C})) = ker\,\delta \cap \Gamma(\mathcal{C})$ *of* δ *in* $\Gamma(\mathcal{C})$ *. With this definition of* σ *and the component-wise definition of* δ^{-1} *(3.7) equations (3.8) and (3.9) hold in* $\Omega(M,\mathcal{C})$ *.*

3. *The Fedosov differential on* $\Omega(M,\mathcal{C})$ *(3.30) can be rewritten in the form*

$$D\mathfrak{P} = \nabla\mathfrak{P} - \delta\mathfrak{P} + \frac{1}{\hbar}[\partial r, \mathfrak{P}]_G \,, \qquad \forall\; \mathfrak{P} \in \Omega(M,\mathcal{C}) \,, \tag{3.36}$$

where r *(3.19) is viewed as an element in* $\Omega^1(M,\mathcal{C}^0)$ *.* $\square$

The following proposition shows that the complex $\Omega^\bullet(M,\mathcal{C}^\bullet)$ is a resolution of the complex $C^\bullet_{loc}(\mathbb{A})$ of local Hochschild cochains of $\mathbb{A}$

Proposition 4 *Fedosov differential (3.30) has vanishing higher cohomology*

$$H^{\geq 1}(\Omega(M,\mathcal{C}^\bullet), D) = 0 \tag{3.37}$$

and

$$H^0(\Omega(M,\mathcal{C}^\bullet), D) = C^\bullet_{loc}(\mathbb{A}) \tag{3.38}$$

as DG associative algebras and as DG Lie algebras.

Proof. To prove (3.37) we pick up $\mathfrak{P} \in \Omega^{\geq 1}(M,\mathcal{C})$ which is closed with respect to the Fedosov differential D

$$D\mathfrak{P} = 0$$

and observe that the recurrent procedure

$$\mathfrak{Q} = -\delta^{-1}\mathfrak{P} + \delta^{-1}(\nabla\mathfrak{Q} + \frac{1}{\hbar}[\partial r, \mathfrak{Q}]_G) \tag{3.39}$$

converges in $2[\hbar] + [y]$-adic topology to an element $\mathfrak{Q} \in \Omega(M,\mathcal{C})$ such that

$$\mathfrak{Q}\Big|_{y=0} = 0 \,. \tag{3.40}$$

Due to equations (3.8), (3.9), extended to $\Omega(M,\mathcal{C})$ by proposition 3, and equation (3.40)

$$\delta^{-1}\mathfrak{Q} = 0\,,$$

and

$$\delta^{-1}(D\mathfrak{Q} - \mathfrak{P}) = 0\,. \tag{3.41}$$

Let us denote $D\mathfrak{Q} - \mathfrak{P}$ by $\mathfrak{K}$. Using (3.8) once again we get that $\mathfrak{K}$ satisfies

$$\mathfrak{K} = \delta^{-1}(\nabla\mathfrak{K} + \frac{1}{\hbar}[\partial r, \mathfrak{K}]_G) \tag{3.42}$$

since $\mathfrak{K} \in \Omega^{\geq 1}(M,\mathcal{C})$ and $D\mathfrak{K} = 0$.

Equation (3.42) has the unique vanishing solution since δ^{-1} raises the degree in y. Hence (3.37) is proven.

To prove the second assertion we mention the following property of the map τ (3.22)

$$\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \tau(a)\Big|_{y^i=0}(x,\hbar) = \partial_{x^{i_1}} \ldots \partial_{x^{i_k}} a(x,\hbar) + \text{ lower order derivatives in } x\,, \tag{3.43}$$

$$\forall\; a \in C^\infty(M)((\hbar))\,.$$

Thus using the map τ we can identify the vector space $C^\bullet_{loc}(\mathbb{A})$ of local Hochschild cochains of $\mathbb{A}$ and the vector space $Z^0_\delta(\Omega(M,\mathcal{C})) = ker\,\delta \cap \Gamma(\mathcal{C})$. An isomorphism from the latter space to the former one is given by the formula

$$(\mu\mathfrak{P})(a_1,\ldots,a_k) = \sigma\mathfrak{P}(\tau(a_1),\ldots,\tau(a_k))\,, \tag{3.44}$$

where $\mathfrak{P} \in Z^0_\delta(\Omega(M,\mathcal{C}))$ and a_i are elements in $C^\infty(M)((\hbar))$. It follows form (3.44) that the star-product (3.23) is the image of μ, namely

$$* = \mu(\circ)\,, \tag{3.45}$$

where $\circ$ is viewed as an element in $\Gamma(\mathcal{C}^2)$ and $* \in C^2_{loc}(\mathbb{A})$.

An appropriate modification of theorem 4 (or theorem 3 in [11]) enables us to conclude that iterating the equation

$$\alpha(\mathfrak{P}) = \mathfrak{P} + \delta^{-1}(\nabla\alpha(\mathfrak{P}) + \frac{1}{\hbar}[\partial r, \alpha(\mathfrak{P})]_G) \tag{3.46}$$

for any $\mathfrak{P} \in Z^0_\delta(\Omega(M,\mathcal{C}))$ one constructs an isomorphism

$$\alpha \;:\; Z^0_\delta(\Omega(M,\mathcal{C})) \xrightarrow{\sim} Z^0_D(\Omega(M,\mathcal{C}))\,,$$

where $Z^0_D(\Omega(M,\mathcal{C})) = ker\, D \cap \Gamma(\mathcal{C})$. It is obvious that

$$\sigma(\alpha(\mathfrak{P})) = \mathfrak{P} \qquad \forall\; \mathfrak{P} \in Z^0_\delta(\Omega(M,\mathcal{C}))\,.$$

Notice that the element $\circ \in \Gamma(\mathcal{C}^2)$ remains unchanged under α

$$\alpha(\circ) = \circ\,.$$

The second claim of the proposition will follow if we prove that the map

$$\beta(\mathfrak{P}) = \mu(\sigma(\mathfrak{P})) \; : \; Z^0_D(\Omega(M, \mathcal{C}^\bullet)) \mapsto C^\bullet_{loc}(\mathbb{A}) \tag{3.47}$$

is an isomorphism of DG associative algebras and DG Lie algebras. We already know that β is an one-to-one and onto. Thus we have to prove the compatibility with the algebraic operations $\cup$, $[,]_G$, and ∂. To do this we observe that for any $k \geq 0$ the map β is given by the formula[3]

$$(\beta\mathfrak{P})(a_1, \dots, a_k) = \sigma(\mathfrak{P}(\tau(a_1), \dots, \tau(a_k)))\,. \tag{3.48}$$

The compatibility with the cup-product (3.27) follows from the line of equations

$$(\beta(\mathfrak{P}_1 \cup \mathfrak{P}_2))(a_1, \dots, a_{k_1+k_2}) = \sigma(\mathfrak{P}_1(\tau(a_1), \dots, \tau(a_{k_1})) \circ \mathfrak{P}_2(\tau(a_{k_1+1}), \dots, \tau(a_{k_1+k_2}))) =$$

$$\sigma(\mathfrak{P}_1(\tau(a_1), \dots, \tau(a_{k_1}))) * \sigma(\mathfrak{P}_2(\tau(a_{k_1+1}), \dots, \tau(a_{k_1+k_2}))) = (\beta\mathfrak{P}_1) \cup (\beta\mathfrak{P}_2)(a_1, \dots, a_{k_1+k_2})\,,$$

where $a_1, \dots, a_{k_1+k_2}$ are arbitrary elements in $\mathbb{A}$, $\mathfrak{P}_1 \in Z^0_D(\Omega(M, \mathcal{C}^{k_1}))$, and $\mathfrak{P}_2 \in Z^0_D(\Omega(M, \mathcal{C}^{k_2}))$.

To prove the compatibility with the Gerstenhaber bracket we observe that for any $a_1, \dots, a_k \in \mathbb{A}$ and $\mathfrak{P} \in Z^0_D(\Omega(M, \mathcal{C}^k))$

$$\tau((\beta\mathfrak{P})(a_1, \dots, a_k)) = \mathfrak{P}(\tau(a_1), \dots, \tau(a_k))\,. \tag{3.49}$$

The latter is proven as follows. Both right and left hand sides of (3.49) are D-closed elements of $\Gamma(\mathcal{W})$. By (3.48) σ of the left hand side equals to σ of the right hand side. Hence (3.49) follows from theorem 4.

Using (3.49) we derive that for any $\mathfrak{P}_1 \in Z^0_D(\Omega(M, \mathcal{C}^{k_1}))$, $\mathfrak{P}_2 \in Z^0_D(\Omega(M, \mathcal{C}^{k_2}))$, and $a_1, \dots, a_{k_1+k_2-1} \in \mathbb{A}$

$$(\beta\mathfrak{P}_1(\dots, \mathfrak{P}_2(\dots), \dots))(a_1, \dots, a_{k_1+k_2-1}) =$$

$$\sigma\mathfrak{P}_1(\tau(a_1), \dots, \mathfrak{P}_2(\tau(a_i), \dots, \tau(a_{i+k_2-1})), \dots, \tau(a_{k_1+k_2-1})) =$$

$$\sigma\mathfrak{P}_1(\tau(a_1), \dots, \tau((\beta\mathfrak{P}_2)(a_i, \dots, a_{i+k_2-1})), \dots, \tau(a_{k_1+k_2-1})) =$$

$$(\beta\mathfrak{P}_1)(a_1, \dots, (\beta\mathfrak{P}_2)(a_i, \dots, a_{i+k_2-1}), \dots, a_{k_1+k_2-1})\,,$$

where $\mathfrak{P}_1(\dots, \mathfrak{P}_2(\dots), \dots)$ denotes the substitution of $\mathfrak{P}_2$ on the place of the i-th argument of $\mathfrak{P}_1$. Thus the compatibility of (3.47) with the Gerstenhaber bracket is proven.

The compatibility of (3.47) with the Hochschild differential follows from the compatibility of (3.47) with the Gerstenhaber bracket, and equations (3.45), (A.7). Thus the proposition is proven. $\square$

Proofs of theorems 1 and 2. It turns out that most of work is already done. Due to theorem 5 of P. Xu [35] we can safely assume that $*$ is the star-product associated to the Fedosov data (M, ∇, Ω_F). Then it is clear that the first claim of theorem 1 would follow from theorem 2.

Proposition 4 implies that the spectral sequence of the double complex $(\Omega^\bullet(M, \mathcal{C}^\bullet), D, \partial)$ associated to the filtration by the exterior degree degenerates at E_1 and the total cohomology of $(\Omega^\bullet(M, \mathcal{C}^\bullet), D, \partial)$

$$H^\bullet(\Omega^\bullet(M, \mathcal{C}^\bullet), D + \partial) = H^\bullet(Z^0_D(\Omega^\bullet(M, \mathcal{C}^\bullet)), \partial)$$

[3] For $k = 0$ the map β just coincides with σ

both as graded associative algebras and as graded Lie algebras. Combining the statements of propositions 2 and 4 we get that $(\Omega^\bullet(M, \mathcal{C}^\bullet), D+\partial, \cup)$ is the desired DG associative algebra $K^\bullet$ and the quasi-isomorphisms in question

$$(\Omega^\bullet(M)((\hbar)), d, \wedge) \xrightarrow{\mathfrak{E}} (\Omega^\bullet(M, \mathcal{C}^\bullet), D+\partial, \cup) \xleftarrow{\mathfrak{D}} (C^\bullet_{loc}(\mathbb{A}), \partial, \cup) \qquad (3.50)$$

are $\mathfrak{E}$ (3.33) and $\mathfrak{D} = \beta^{-1}$ (3.47). The naturality of $K^\bullet$, $\mathfrak{D}$ and $\mathfrak{E}$ with respect to the Fedosov data is evident.

While the equivariance property (2.6) of $\mathfrak{E}$ is obvious from the construction, the equivariance property (2.6) of $\mathfrak{D}$ follows essentially from the fact that any diffeomorphism acts on the fiber variables y^i of the tangent bundle TM by linear transformations. Thus theorem 2 is proven.

To prove the statement about the Gerstenhaber bracket in theorem 1 we observe that due to propositions 8 and 9 in Appendix B any cocycle in $(\Omega^\bullet(M, \mathcal{C}^\bullet), D+\partial)$ is cohomologically equivalent to a cocycle in $\Gamma(\mathcal{C}^0))$. But the restriction of the Gerstenhaber bracket to $\mathcal{C}^0$ is vanishing by definition (A.6). This completes the proofs of both theorems. $\square$

4 Concluding remarks

In this section we discuss applications and variations of theorems 1 and 2 .

First, using theorems 1 and 2 , one can easily prove an equivariant version of Xu's theorem [35]

Corollary 1 *If G is a group acting on M by symplectomorphisms and M admits a G-invariant connection ∇ then any G-invariant star-product is equivalent to some G-invariant Fedosov star-product. If G is finite or compact then the equivalence can be established by a G-invariant operator.* $\square$

Second, in some cases it is instructive to know the Hochschild cohomology of the algebra $\mathbb{A}_0 = C^\infty(M)[[\hbar]]$ of formal Taylor power series with multiplication $*$. By keeping track of negative degrees in $\hbar$ in our construction one can easily prove that

Proposition 5 *The graded associative algebras*

$$(HH^\bullet_{loc}(\mathbb{A}_0), \cup)$$

and

$$H^\bullet(\Omega(M)[[\hbar]], \hbar\, d, \wedge)$$

are isomorphic. $\square$

Third, natural algebraic geometric versions of theorems 1 and 2 hold for a smooth affine algebraic variety X (over $\mathbb{C}$). These versions immediately follow from Grothendieck's theorem on De Rham cohomology of an affine variety and the fact that any smooth affine algebraic variety admits an algebraic connection on the holomorphic tangent bundle $T_{hol}X$.

Proposition 6 *If X is a smooth affine algebraic variety over $\mathbb{C}$ endowed with an algebraic symplectic form ω, and $\mathbb{A} = (\mathcal{O}(X)((\hbar)), *)$ is the corresponding quantum ring of functions then the graded associative algebra*

$$(HH^{\bullet}(\mathbb{A}), \cup)$$

of Hochschild cohomology of $\mathbb{A}$ is isomorphic to the graded associative algebra

$$(H^{\bullet}_{DR}(X), \cup)$$

of De Rham cohomology of X. The Gerstenhaber bracket on $HH^{\bullet}(\mathbb{A})$ is vanishing. $\square$

It is worth mentioning that in this algebraic geometric case the complex of local Hochschild cochains is quasi-isomorphic to the complex of all Hochschild cochains. For this reason, the analogue of theorem 1 is formulated for the genuine Hochschild cohomology of the quantum ring $\mathbb{A}$.

Notice that using the latter proposition only for the Fedosov star-products and rearranging the arguments, one can prove an algebraic geometric version of Xu's theorem [35] (theorem 5) for any smooth symplectic affine algebraic variety over $\mathbb{C}$.

Finally, we would like to mention paper [15] in which the authors propose an explicit expression for the canonical trace in deformation quantization of a symplectic manifold. We suspect that their formula for the trace can be obtained with the help of quasi-isomorphisms between the bar resolution and the Koszul resolution of Weyl algebra (see proposition 7) and the origin of the integrals over the configuration space of ordered points on a circle in their formula is the result of multiple applications of a contracting operator similar to (B.23).

Acknowledgment. This paper arose from questions of my advisor Pavel Etingof to whom I express my sincere thanks. I also acknowledge Pavel for his valuable criticisms concerning the first version of the manuscript. I would like to thank Alexander Braverman, Simone Gutt, Lars Hesselholt, Richard Melrose, Alexei Oblomkov, and Dmitry Tamarkin for useful discussions. I am grateful to D. Silinskaia for criticisms concerning my English. The work is partially supported by the NSF grant DMS-9988796, the Grant for Support of Scientific Schools NSh-1999.2003.2, the grant INTAS 00-561 and the grant CRDF RM1-2545-MO-03.

5 Appendix A. Algebraic structures on Hochschild cochains.

Let A be an associative unital algebra over a field of characteristic zero. By definition, Hochschild cohomology [24] of A is

$$HH^{\bullet}(A) = Ext^{\bullet}_{A\otimes A^{op}}(A, A)\,, \tag{A.1}$$

where A^{op} is the algebra A with the opposite multiplication and A is naturally viewed as a left module over $A \otimes A^{op}$.

Using the standard bar resolution for A one shows that Hochschild cohomology (A.1) is cohomology of the following complex

$$C^m(A) = Hom(A^{\otimes m}, A)\,,\ (m \geq 1)\,, \qquad C^0(A) = A \tag{A.2}$$

with the differential given by

$$\partial\Phi(a_1,\dots a_{m+1}) = a_1\cdot\Phi(a_2,\dots,a_{m+1}) - \Phi(a_1\cdot a_2, a_3,\dots,a_{m+1}) + \dots$$

$$+(-)^m\Phi(a_1,\dots,a_{m-1},a_m\cdot a_{m+1}) + (-)^{m+1}\Phi(a_1,\dots,a_{m-1},a_m)\cdot a_{m+1}\,, \qquad (A.3)$$

$$\partial\,:\,C^m(A)\mapsto C^{m+1}(A)\,.$$

The vector space (A.2) is usually referred to as the space of Hochschild cochains of the associative algebra A.

Using the product in the algebra A one can define the associative cup product of Hochschild cochains. The cup product of two homogeneous cochains $\Phi_1\in C^{k_1}(A)$ and $\Phi_2\in C^{k_2}(A)$ is given by the formula

$$\Phi_1\cup\Phi_2(a_1,a_2,\dots,a_{k_1+k_2}) = \Phi_1(a_1,\dots,a_{k_1})\cdot\Phi_2(a_{k_1+1},\dots,a_{k_1+k_2})\,, \qquad (A.4)$$

where $a_i\in A$. It is not hard to show that the Hochschild differential (A.3) is a derivation of the cup product (A.4)

$$\partial(\Phi_1\cup\Phi_2) = (\partial\Phi_1)\cup\Phi_2 + (-)^{k_1}\Phi_1\cup(\partial\Phi_2) \qquad (A.5)$$

and the induced product on the cohomology coincides with the Yoneda product [17] in (A.1).

The space $C^\bullet(A)$ can be also endowed with the so-called Gerstenhaber bracket [19] which is defined between homogeneous elements $\Phi_1\in C^{k_1+1}(A)$ and $\Phi_2\in C^{k_2+1}(A)$ as follows

$$[\Phi_1,\Phi_2]_G(a_0,\,\dots,a_{k_1+k_2}) =$$

$$\sum_{i=0}^{k_1}(-)^{ik_2}\Phi_1(a_0,\,\dots,\Phi_2(a_i,\,\dots,a_{i+k_2}),\,\dots,a_{k_1+k_2}) \qquad (A.6)$$

$$-(-)^{k_1k_2}(1\leftrightarrow 2)\,,\qquad a_j\in A\,.$$

Direct computation shows that (A.6) determines a Lie (super)bracket on the space $C^\bullet(A)[1]$ of the Hochschild cochains with a shifted grading. One can observe that the differential (A.3) can be rewritten in terms of the bracket (A.6) as follows

$$\partial\Phi = (-)^{k+1}[\mu_0,\Phi]_G : C^k(A)\mapsto C^{k+1}(A)\,, \qquad (A.7)$$

where $\mu_0\in C^2(A)$ is the multiplication in the algebra A. This observation implies that ∂ is a derivation of the Gerstenhaber bracket (A.6)

$$\partial[\Phi_1,\Phi_2]_G = [\partial\Phi_1,\Phi_2]_G + (-)^{k_1-1}[\Phi_1,\partial\Phi_2]_G\,,\qquad \Phi_i\in C^{k_i}(A)\,. \qquad (A.8)$$

6 Appendix B. Equivariant resolution of the Weyl algebra.

In this section we propose a $GL(2n,\mathbb{C})$-equivariant homotopy formula for the cohomological Hochschild complex of the (formal) Weyl algebra[4].

Let θ^{ij} be a non-degenerate antisymmetric matrix of size $2n\times 2n$. Then the vector space of the Weyl algebra W is by definition the vector space $\mathbb{C}[[y^1,\dots,y^{2n}]]((\hbar))$ of formal Laurent power series in $\hbar$ whose coefficients are formal Taylor power series in $y^1,\dots,y^{2n}$. The multiplication $\circ$ in W is given by

$$a\circ b=\exp\left(\frac{\hbar}{2}\theta^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(\hbar,y)\,b(\hbar,z)|_{z=y}\,,\qquad a,b\in W\,. \tag{B.1}$$

One can observe that W is naturally filtered with respect to the degree of monomials $2[\hbar]+[y]$ where $[\hbar]$ is a degree in $\hbar$ and $[y]$ is the total degree in y's

$$\dots\subset W^1\subset W^0\subset W^{-1}\dots\subset W\,,\quad W^m=\{a\in W\,|\,a=\sum_{2k+p\geq m}\hbar^k a_{k;i_1\dots i_p}y^{i_1}\dots y^{i_p}\}\,. \tag{B.2}$$

This filtration defines the $2[\hbar]+[y]$-adic topology on the algebra W and the product (B.1) is continuous in this topology.

Since W is a topological algebra one should be careful with the standard arguments of homological algebra. In particular, the definition of Hochschild cohomology for the algebra should be slightly modified. By definition,

$$HH^k(W)=H^k(C^\bullet(W),\partial)\,,\qquad k\geq 0\,, \tag{B.3}$$

where $C^q(W)$ is the vector space of continuous $\mathbb{C}((\hbar))$-linear maps

$$\Phi\in Hom_{\mathbb{C}((\hbar))}(W^{\tilde{\otimes}q},W)\,,$$

and $\tilde{\otimes}$ stands for the tensor product over $\mathbb{C}((\hbar))$ completed in topology (B.2). Any such map can be uniquely represented in the form of the following formal series

$$\Phi=\sum_m\sum_{\alpha_1\dots\alpha_q}\sum_{p=0}^{\infty}\hbar^m\Phi^{\alpha_1\dots\alpha_q}_{m,i_1\dots i_p}y^{i_1}\dots y^{i_p}\frac{\partial}{\partial y^{\alpha_1}}\otimes\dots\otimes\frac{\partial}{\partial y^{\alpha_q}}\,, \tag{B.4}$$

where α's are multi-indices $\alpha=(j_1\dots j_l)$,

$$\frac{\partial}{\partial y^\alpha}=\frac{\partial}{\partial y^{j_1}}\dots\frac{\partial}{\partial y^{j_l}}\,,$$

and the summation in m is bounded below.

It is not hard to see that the Hochschild differential ∂ (A.3), the cup-product (A.4) and the Gerstenhaber bracket (A.6) still make sense for the complex $C^\bullet(W)$.

[4]See paper [1] in which a similar computation has been performed for the algebra of invariants of the ordinary (non-formal) Weyl algebra acted upon by a finite group of automorphisms.

Similarly, we replace the “stupid” bar resolution by the topological bar resolution of W as a left $W\widetilde{\otimes}W^{op}$-module

$$B=\bigoplus_{m=0}^{\infty}B_m\,,\qquad B_m=W^{\widetilde{\otimes}(m+2)}=\mathbb{C}[[\tilde{y}_1,\ldots,\tilde{y}_{m+2}]]((\hbar))\,, \tag{B.5}$$

where $\tilde{y}_i=(y_i^1,\ldots,y_i^{2n})$, W^{op} denotes the Weyl algebra with the opposite multiplication, the differential $\mathfrak{b}\,:\,B_m\mapsto B_{m-1}$ is given by the formula

$$(\mathfrak{b}b)(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+1})=$$

$$\sum_{k=1}^{m}(-)^{k-1}\exp(\frac{\hbar}{2}\theta^{ij}\frac{\partial}{\partial y_k^i}\frac{\partial}{\partial z^j})b(\hbar,\tilde{y}_1,\ldots,\tilde{y}_k,\tilde{z},\tilde{y}_{k+1},\ldots,\tilde{y}_{m+1})\Big|_{\tilde{z}=\tilde{y}_k}\,, \tag{B.6}$$

and the $W\widetilde{\otimes}W^{op}$-module structure is defined by

$$\begin{gathered}(a\cdot b)(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+2})=\exp(\frac{\hbar}{2}\theta^{ij}\frac{\partial}{\partial z^i}\frac{\partial}{\partial y_1^j})a(\hbar,\tilde{z})b(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+2})\Big|_{\tilde{z}=\tilde{y}_1}\,,\\ (b\cdot a)(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+2})=\exp(\frac{\hbar}{2}\theta^{ij}\frac{\partial}{\partial y_{m+2}^i}\frac{\partial}{\partial z^j})b(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+2})a(\hbar,\tilde{z})\Big|_{\tilde{z}=\tilde{y}_{m+2}}\,,\\ a\in W,\qquad b\in\mathbb{C}[[\tilde{y}_1,\ldots,\tilde{y}_{m+2}]]((\hbar))\,.\end{gathered} \tag{B.7}$$

By h_B we denote the corresponding homotopy operator $h_B\,:\,B_{m-1}\mapsto B_m$

$$(h_Bb)(\hbar,\tilde{y}_1,\ldots,\tilde{y}_{m+2})=b(\hbar,\tilde{y}_2,\ldots,\tilde{y}_{m+2}) \tag{B.8}$$

in the augmented bar complex

$$\cdots\stackrel{\mathfrak{b}}{\to}B_2\stackrel{\mathfrak{b}}{\to}B_1\stackrel{\mathfrak{b}}{\to}B_0\stackrel{\circ}{\to}W\,. \tag{B.9}$$

Obviously, we have the following canonical isomorphism of complexes

$$C^{\bullet}(W)=Hom^c_{W\widetilde{\otimes}W^{op}}(B_{\bullet},W)\,, \tag{B.10}$$

where by Hom^c we denote the vector space of all homomorphisms continuous in topology (B.2).

An appropriate analogue of the Koszul resolution of the Weyl algebra W is the polynomial algebra

$$K=W\widetilde{\otimes}W^{op}[C^1,\ldots,C^{2n}] \tag{B.11}$$

in $2n$ *anticommuting* variables $C^1,\ldots,C^{2n}$ with coefficients in the tensor product $W\widetilde{\otimes}W^{op}$ over $\mathbb{C}((\hbar))$ completed in topology (B.2). The differential in K is given by the formula

$$\mathfrak{d}a=(y_1^i-y_2^i)\frac{\vec{\partial}}{\partial C^i}a+\frac{\hbar}{2}\theta^{ij}\frac{\vec{\partial}}{\partial C^i}(\frac{\partial}{\partial y_1^i}+\frac{\partial}{\partial y_2^i})a\,,\qquad a\in K\,, \tag{B.12}$$

where the arrow over ∂ means that we use left derivatives with respect to anticommuting variables C^i.

The m-th term of the complex (B.11)

$$K_m = \wedge^m(\mathbb{C}^{2n}) \otimes W\widetilde{\otimes}W^{op}$$

is naturally a left module over $W\widetilde{\otimes}W^{op}$ and the differential (B.12) $\mathfrak{d} : K_m \mapsto K_{m-1}$ is compatible with the $W\widetilde{\otimes}W^{op}$-module structure.

The homotopy operator h_K of the augmented Koszul complex

$$\dots \stackrel{\mathfrak{d}}{\to} \wedge^m(\mathbb{C}^{2n}) \otimes W\widetilde{\otimes}W^{op} \stackrel{\mathfrak{d}}{\to} \dots \stackrel{\mathfrak{d}}{\to} \mathbb{C}^{2n} \otimes W\widetilde{\otimes}W^{op} \stackrel{\mathfrak{d}}{\to} W\widetilde{\otimes}W^{op} \stackrel{\circ}{\to} W \tag{B.13}$$

looks as follows

$$\begin{aligned} h_K(a) = C^k \int_0^1 dt (\mathcal{D}_{-t}\, \mathcal{D}\, \frac{\partial}{\partial y_1^k} a)((\hbar, \tilde{y}_2 + t(\tilde{y}_1 - \tilde{y}_2), \tilde{y}_2, tC))\,, \\ \mathcal{D} = \exp\left(\frac{\hbar}{2}\theta^{ij}\frac{\partial}{\partial y_1^i}\frac{\partial}{\partial y_2^j}\right)\,, \\ \mathcal{D}_{-t} = \exp\left(-\frac{\hbar t}{2}\theta^{ij}\frac{\partial}{\partial y_1^i}\frac{\partial}{\partial y_2^j}\right)\,. \end{aligned} \tag{B.14}$$

Using homotopy operators (B.8), (B.14) we can inductively construct continuous quasi-isomorphisms between the topological bar resolution (B.5) and the Koszul resolution (B.11). Namely,

Proposition 7 *The complexes of left $W\widetilde{\otimes}W^{op}$-modules (B.5) and (B.11) are quasi-isomorphic. A quasi-isomorphism λ from the Koszul resolution (B.11) to the topological bar resolution (B.5) is determined inductively by setting*

$$\lambda\Big|_{W\widetilde{\otimes}W^{op}} = id \,:\, K_0 \mapsto B_0\,,$$

and

$$\lambda(a) = h_B(\lambda(\mathfrak{d}a))$$

for any topological generator a of the left $W\widetilde{\otimes}W^{op}$-module K_m $(m>0)$. A quasi-isomorphism ν from the topological bar resolution (B.5) to the Koszul resolution (B.11) is determined inductively by setting

$$\nu\Big|_{W\widetilde{\otimes}W^{op}} = id \,:\, B_0 \mapsto K_0\,,$$

and

$$\nu(b) = h_K(\nu(\mathfrak{b}b))$$

for any topological generator b of the left $W\widetilde{\otimes}W^{op}$-module B_m $(m>0)$.

Proof. We define inductively a morphism $\rho : B \mapsto B[1]$ of left $W\widetilde{\otimes}W^{op}$-modules by setting

$$\rho\Big|_{W\widetilde{\otimes}W^{op}} = 0 \,:\, B_0 \mapsto B_1\,, \qquad \rho(b) = h_B(id - \lambda\nu)(b) - h_B\rho(\mathfrak{b}(b)) \tag{B.15}$$

for any topological generator b of the left $W\widetilde{\otimes}W^{op}$-module B_m $(m>0)$. Direct computation shows that ρ is a homotopy operator between id and $\lambda\,\nu$

$$b - \lambda(\nu(b)) = \mathfrak{b}(\rho(b)) + \rho(\mathfrak{b}(b))\,. \tag{B.16}$$

Similarly, one constructs a homotopy operator between id and $\nu\,\lambda$ on K $\square$.

The above proposition immediately implies that the maps

$$(\hat{\lambda}a)(\kappa) = a(\lambda\kappa) \,:\, C^\bullet(W) \mapsto Hom^c_{W\tilde{\otimes}W^{op}}(K_\bullet, W)\,, \qquad \kappa \in K_\bullet\,,\ a \in C^\bullet(W) \tag{B.17}$$

$$(\hat{\nu}f)(b) = f(\nu b) \,:\, Hom^c_{W\tilde{\otimes}W^{op}}(K_\bullet, W) \mapsto C^\bullet(W)\,, \tag{B.18}$$
$$b \in B_\bullet\,, \qquad f \in Hom^c_{W\tilde{\otimes}W^{op}}(K_\bullet, W)$$

are quasi-isomorphisms of the complexes $C^\bullet(W)$ and $Hom^c_{W\tilde{\otimes}W^{op}}(K_\bullet, W)$ and the map

$$(\hat{\rho}a)(b) = a(\rho b) \,:\, C^\bullet(W) \mapsto C^\bullet(W)\,, \qquad a \in C^\bullet(W)\,,\ b \in B_\bullet \tag{B.19}$$

is a homotopy operator between the id and $\hat{\nu}\,\hat{\lambda}$

$$a - \hat{\nu}(\hat{\lambda}\,a) = (\partial\hat{\rho} + \hat{\rho}\partial)a\,, \qquad \forall\ a \in C^\bullet(W)\,. \tag{B.20}$$

In the above equations we use the identification of $C^\bullet(W)$ and $Hom^c_{W\tilde{\otimes}W^{op}}(B_\bullet, W)$ (B.10).

The complex $Hom^c_{W\tilde{\otimes}W^{op}}(K_\bullet, W)$ is obviously identified with the polynomial algebra $W[\psi_1, \dots, \psi_{2n}]$ in $2n$ *anticommuting* variables $\psi_1, \dots, \psi_{2n}$ with coefficients in the Weyl algebra W, namely

$$Hom^c_{W\tilde{\otimes}W^{op}}(K_q, W) = W[\psi_1, \dots, \psi_{2n}]_{(q)}, \tag{B.21}$$

where $W[\psi_1, \dots, \psi_{2n}]_{(q)}$ is the space of polynomials in ψ's of degree q. One can easily compute the differential ∂_H in $W[\psi_1, \dots, \psi_{2n}]$

$$\partial_H a = \hbar\psi_i\omega^{ij}\frac{\partial}{\partial y^j}a\,, \qquad a \in W[\psi_1, \dots, \psi_{2n}], \tag{B.22}$$

and guess the corresponding partial homotopy operator

$$\mathcal{H}a = \frac{1}{\hbar}\int_0^1 dt\, y^i\omega_{ij}(\frac{\vec{\partial}}{\partial\psi_j}a)(\hbar, ty, t\psi)\,, \qquad a \in W[\psi_1, \dots, \psi_{2n}] \tag{B.23}$$

satisfying the following identity

$$a = a\Big|_{y=\psi=0} + (\partial_H\mathcal{H} + \mathcal{H}\partial_H)a\,, \qquad \forall\ a \in W[\psi_1, \dots, \psi_{2n}]\,. \tag{B.24}$$

Thus we conclude that

$$HH^q(W) = \begin{cases} \mathbb{C}((\hbar))\,,\ q = 0\,, \\ 0\,,\ \text{otherwise}\,. \end{cases} \tag{B.25}$$

For the case $q = 0$ we also have an obvious algebraic formulation of the above assertion

$$Z^0(C^\bullet(W), \partial) = Z(W) = \mathbb{C}((\hbar))\,, \tag{B.26}$$

where $Z^0(C^\bullet(W), \partial)$ denotes the space of the zeroth cocycles of the complex $C^\bullet(W)$ and $Z(W)$ stands for the center of the Weyl algebra W.

Together with the expected result (B.25) the above considerations enable us to get a homotopy formula for the Hochschild cohomological complex of the Weyl algebra. Using proposition 7 and equations (B.20), (B.24) we derive that

Proposition 8 *The following operator*

$$\chi(a) = \hat{\nu}(\mathcal{H}(\hat{\lambda}(a))) + \hat{\rho}(a)\,, \qquad a \in C^{\geq 1}(W) \tag{B.27}$$

satisfies the equation

$$a = (\partial\chi + \chi\partial)a \tag{B.28}$$

for any $a \in C^{\geq 1}(W)$. $\square$

An explicit expression of the homotopy operator (B.27) is rather complicated. See, for example, paper [21] in which an explicit homotopy formula is given for the Hochschild complexes of the commutative algebra of polynomials in n variables.

One can observe that any element $g \in GL(2n, \mathbb{C})$ defines an isomorphism from the Weyl algebra W_θ associated to the (antisymmetric, non-degenerate) matrix θ to the Weyl algebra $W_{\theta'}$ associated to the matrix $\theta' = g_*\theta$. For our purposes we need to keep track of the equivariance properties with respect to such isomorphisms. To do this we introduce a groupoid of Weyl algebras. The objects of this groupoid are Weyl algebras W_θ associated to all possible antisymmetric non-degenerate matrices θ and the morphisms are defined by elements of $GL(2n, \mathbb{C})$. One can repeat all the arguments in this appendix by replacing a single Weyl algebra by this groupoid. Thus we get the following

Proposition 9 *The homotopy operator (B.27) is* $GL(2n, \mathbb{C})$*-equivariant. Namely, if* χ_θ *denotes the homotopy operator (B.27) in the complex* $C^\bullet(W_\theta)$ *then for any* $g \in GL(2n, \mathbb{C})$ *the diagram*

$$\begin{array}{ccc} C^\bullet(W_\theta) & \stackrel{\chi_\theta}{\to} & C^{\bullet-1}(W_\theta) \\ \downarrow^{g_*} & & \downarrow^{g_*} \\ C^\bullet(W_{g_*\theta}) & \stackrel{\chi_{g_*\theta}}{\to} & C^{\bullet-1}(W_{g_*\theta}) \end{array} \tag{B.29}$$

commutes. $\square$

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